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Error estimates for some composite corrected quadrature rules

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ABSTRACT

The asymptotic behaviour of the error for a general quadrature rule is established and it is applied to some composite corrected quadrature rules.

1. Introduction

Let $f : [a, b] \to \mathbf{R}$ be such that f' is absolutely continuous on $[a, b]$. It is well known that the trapezoidal rule

$$T(f) := \frac{b-a}{2}[f(a) + f(b)] \tag{1}$$

and the midpoint rule

$$M(f) := (b-a)f\left(\frac{a+b}{2}\right) \tag{2}$$

are the simplest quadrature formulae used to approximate the integral

$$\int_a^b f(t)\mathrm{d}t$$

which can serve as basic elements for constructing more sophisticated formulae by certain types of convex combinations, e.g., the classical Simpson rule is defined as

$$S(f) = \frac{1}{3}T(f) + \frac{2}{3}M(F) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{3}$$

Further, we may consider the corrected or perturbed trapezoidal rule

$$CT(f) := \frac{b-a}{2}[f(a) + f(b)] - \frac{(b-a)^2}{12}[f'(b) - f'(a)] \tag{4}$$

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and the corrected or perturbed midpoint rule

$$CM(f) := (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24}[f'(b) - f'(a)]. \tag{5}$$

It is well known that corrected quadrature formulae (4) and (5) have better estimates of error than corresponding original formulae (1) and (2). (See, e.g., [1–3].)

Unfortunately, we could not obtain such a corrected version of the Simpson rule, since

$$\frac{1}{3}CT(f) + \frac{2}{3}CM(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

does not involve the derivatives at the endpoints.

Nevertheless, we did find in [4–6] the so-called corrected Simpson quadrature rule that improves on Simpson rule (3):

$$\begin{aligned} CS(f) &= \frac{7}{15}CT(f) + \frac{8}{15}CM(f) \\ &= \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] - \frac{(b-a)^2}{60}[f'(b) - f'(a)]. \end{aligned} \tag{6}$$

It is not difficult to find that the corrected trapezoidal rule (4) and the corrected midpoint rule (5) are exact for polynomials of degree 3 or less, and the corrected Simpson rule (6) is exact for polynomials of degree 5 or less.

Now let $a = t_0 < t_1 < \cdots < t_n = b$ be an equidistant subdivision of the interval $[a, b]$ such that $t_{i+1} - t_i = h = \frac{b-a}{n}$, $i = 0, 1, \ldots, n-1$. Then we have the composite corrected trapezoidal rule

$$CT_n(f) = \frac{b-a}{2n}[f(t_0) + 2f(t_1) + \cdots + 2f(t_{n-1}) + f(t_n)] - \frac{(b-a)^2}{12n^2}[f'(b) - f'(a)], \tag{7}$$

the composite corrected midpoint rule

$$CM_n(f) = \frac{b-a}{n}\left[f\left(\frac{t_0+t_1}{2}\right) + f\left(\frac{t_1+t_2}{2}\right) + \cdots + f\left(\frac{t_{n-1}+t_n}{2}\right)\right] + \frac{(b-a)^2}{24n^2}[f'(b) - f'(a)] \tag{8}$$

and the composite corrected Simpson rule

$$\begin{aligned} CS_n(f) = \frac{b-a}{30n}\Bigg\{&7[f(t_0) + 2f(t_1) + \cdots + 2f(t_{n-1}) + f(t_n)] \\ &+ 16\left[f\left(\frac{t_0+t_1}{2}\right) + f\left(\frac{t_1+t_2}{2}\right) + \cdots + f\left(\frac{t_{n-1}+t_n}{2}\right)\right]\Bigg\} - \frac{(b-a)^2}{60n^2}[f'(b) - f'(a)] \end{aligned} \tag{9}$$

which correspond to the corrected quadrature rules (4)–(6), respectively.

Motivated by [7,8], in this work, we will give a unified treatment for estimating the errors for the above mentioned composite corrected quadrature rules (7)–(9).

2. Main results

We need the following result (see Theorem 10 in [8] or Theorem 3 in [9]):

Lemma 1. *Let $f : [a, b] \to \mathbf{R}$ be such that its $(r-1)$th derivative $f^{(r-1)}$ is of continuous bounded variation for some positive integer r. Then for any $x \in [0, 1]$, we have*

$$\int_a^b f(t)\mathrm{d}t = h\sum_{i=0}^{n-1} f(t_i + xh) - \sum_{\nu=1}^{r} \frac{f^{(\nu-1)}(b) - f^{(\nu-1)}(a)}{\nu!} B_\nu(x)h^\nu + R_n^{(r)}(f; x), \tag{10}$$

where

$$R_n^{(r)}(f; x) = \frac{h^r}{r!}\int_a^b \tilde{B}_r\left(x - n\frac{t-a}{b-a}\right)\mathrm{d}f^{(r-1)}(t) \tag{11}$$

and $\tilde{B}_r(t) := B_r(t - [t])$ while $B_r(t)$ is the rth Bernoulli polynomial.

Theorem 2. *Suppose that for some real constants $\alpha \neq 0$ and $p_j \in [0, 1]$, $j = 0, 1, \dots, m-1$, with $\sum_{j=0}^{m-1} p_j = 1$, the following quadrature rule*

$$\int_0^1 f(t)\mathrm{d}t = \sum_{j=0}^{m-1} p_j f(x_j) + \alpha[f'(1) - f'(0)] \tag{12}$$

is exact for any polynomial of degree $\leq r-1$ for some positive integer $r \geq 3$. Let $f : [a, b] \to \mathbf{R}$ be such that its $(r-1)$th derivative $f^{(r-1)}$ is a continuous function of bounded variation on $[a, b]$. Then we have

$$\int_a^b f(t)\mathrm{d}t = h \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} p_j f(t_i + x_j h) + \alpha h^2[f'(b) - f'(a)] + R_n^{(r)}(f), \tag{13}$$

where

$$R_n^{(r)}(f) = h^r \int_a^b G_r\left(n\frac{t-a}{b-a}\right) \mathrm{d}f^{(r-1)}(t), \tag{14}$$

and

$$G_r(t) = \frac{1}{r!} \sum_{j=0}^{m-1} p_j(\tilde{B}_r(x_j - t) - B_r(x_j)). \tag{15}$$

Proof. We first note that for $1 \leq \nu \leq r-1$,

$$\sum_{j=0}^{m-1} p_j B_\nu(x_j) + \alpha[B_\nu'(1) - B_\nu'(0)] = \int_0^1 B_\nu(t)\mathrm{d}t = 0$$

due to (12) being exact for any polynomial of degree $\leq r-1$ and a well known property of the Bernoulli polynomials. Moreover, since $B_0(t) = 1$, $B_n'(t) = nB_{n-1}(t)$ and $B_n(t+1) - B_n(t) = nt^{n-1}$, $n = 1, 2, \dots$, we get

$$\sum_{j=0}^{m-1} p_j B_\nu(x_j) = \begin{cases} 0, & \text{if } 1 \leq \nu \leq r-1,\ \nu \neq 2, \\ -2\alpha, & \text{if } \nu = 2. \end{cases} \tag{16}$$

Now setting $x = x_j$ in (10), multiplying both sides of (10) by p_j, summing from $j = 0$ to $m-1$, from (10), (16) and the following readily checked fact:

$$\frac{f^{(r-1)}(b) - f^{(r-1)}(a)}{r!} B_r(x) h^r = \frac{h^r}{r!} \int_a^b B_r(x) \mathrm{d}f^{(r-1)}(t),$$

we obtain

$$\int_a^b f(t)\mathrm{d}t = h \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} p_j f(t_i + x_j h) - \sum_{\nu=1}^{r} \left(\sum_{j=0}^{m-1} p_j B_\nu(x_j)\right) \frac{f^{(\nu-1)}(b) - f^{(\nu-1)}(a)}{\nu!} h^\nu + \sum_{j=0}^{m-1} p_j R_n^{(r)}(f; x_j),$$

i.e.,

$$\begin{aligned}\int_a^b f(t)\mathrm{d}t &= h \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} p_j f(t_i + x_j h) - \sum_{\nu=1}^{r-1} \left(\sum_{j=0}^{m-1} p_j B_\nu(x_j)\right) \frac{f^{(\nu-1)}(b) - f^{(\nu-1)}(a)}{\nu!} h^\nu \\ &\quad - \sum_{j=0}^{m-1} p_j B_r(x_j) \frac{f^{(r-1)}(b) - f^{(r-1)}(a)}{r!} h^r + \sum_{j=0}^{m-1} p_j R_n^{(r)}(f; x_j),\end{aligned}$$

and it follows from (16) that

$$\begin{aligned}\int_a^b f(t)\mathrm{d}t &= h \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} p_j f(t_i + x_j h) + \alpha h^2[f'(b) - f'(a)] \\ &\quad - \sum_{j=0}^{m-1} p_j \frac{h^r}{r!} \int_a^b B_r(x_j) \mathrm{d}f^{(r-1)}(t) + \sum_{j=0}^{m-1} p_j R_n^{(r)}(f; x_j).\end{aligned}$$

Finally, from (11) we get

$$\int_a^b f(t)\mathrm{d}t = h\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} p_j f(t_i + x_j h) + \alpha h^2[f'(b) - f'(a)] + \frac{h^r}{r!}\int_a^b \sum_{j=0}^{m-1} p_j\left(\tilde{B}_r\left(x_j - n\frac{t-a}{b-a}\right) - B_r(x_j)\right)\mathrm{d}f^{(r-1)}(t),$$

which completes the proof. □

Remark 3. The case $\alpha = 0$ has already been studied in [8]. For $\alpha \neq 0$, from the proof of Theorem 2, we can conclude that

$$\sum_{\nu=1}^{r-1}\left(\sum_{j=0}^{m-1} p_j B_\nu(x_j)\right)\frac{f^{(\nu-1)}(b) - f^{(\nu-1)}(a)}{\nu!}h^\nu = -\alpha h^2[f'(b) - f'(a)]$$

only if $r \geq 3$. Otherwise (for $r = 1, 2$) the expression on the left-hand side equals 0.

Theorem 4. *Let the assumptions of Theorem 2 hold. Then we have*

$$|n^r R_n^{(r)}(f)| \leq C_r (b-a)^r \bigvee_a^b (f^{(r-1)}), \tag{17}$$

where

$$C_r := \frac{1}{r!}\sup_{0<t<1}\left|\sum_{j=0}^{m-1} p_j(\tilde{B}_r(x_j - t) - B_r(x_j))\right|. \tag{18}$$

If further $f^{(r-1)}$ is absolutely continuous on $[a, b]$, we then have

$$\lim_{n\to\infty} n^r R_n^{(r)}(f) = K_r (b-a)^r \int_a^b f^{(r)}(t)\mathrm{d}t, \tag{19}$$

where

$$K_r := -\frac{1}{r!}\sum_{j=0}^{m-1} p_j B_r(x_j). \tag{20}$$

The proof is similar to that for the Theorem 12 in [8] and so is omitted.

Remark 5. If (12) has degree of precision $r - 1$ then C_ν and K_ν exist for all $3 \leq \nu \leq r$ since in this case (12) is exact for any polynomial of degree $\leq \nu - 1$. We obtain from (16) and (20) that

$$K_\nu = 0, \quad \text{if } 3 \leq \nu < r, \tag{21}$$

which implies from (19) that

$$\lim_{n\to\infty} n^\nu R_n^{(\nu)}(f) = 0, \quad \text{if } 3 \leq \nu < r.$$

3. Examples

Example 1. For the corrected trapezoid rule, we see that $\alpha = -\frac{1}{12}$ and it has degree of precision 3 ($r = 4$). Thus for any function f such that $f^{(\nu-1)}$ ($\nu = 3, 4$) is continuous of bounded variation, we get C_ν by direct calculations from (18) as

$$C_3 = \frac{\sqrt{3}}{216}, \qquad C_4 = \frac{1}{384},$$

and, for any function f such that $f^{(\nu-1)}$ ($\nu = 3, 4$) is absolutely continuous, we obtain K_ν from (21) and by a direct calculation from (20) as

$$K_3 = 0, \qquad K_4 = \frac{1}{720}$$

and these imply that

$$\lim_{n\to\infty} n^3 R(f; CT_n) = \lim_{n\to\infty} n^3 R_n^{(3)}(f) = 0$$

and

$$\lim_{n\to\infty} n^4 R(f; CT_n) = \lim_{n\to\infty} n^4 R_n^{(4)}(f) = \frac{1}{720}(b-a)^4[f'''(b) - f'''(a)].$$

Example 2. For the corrected midpoint rule, we see that $\alpha = \frac{1}{24}$ and it has degree of precision 3 ($r = 4$). Thus for any function f such that $f^{(\nu-1)}$ ($\nu = 3, 4$) is continuous of bounded variation, we get C_ν by direct calculations from (18) as

$$C_3 = \frac{\sqrt{3}}{216}, \qquad C_4 = \frac{1}{384},$$

and, for any function f such that $f^{(\nu-1)}$ ($\nu = 3, 4$) is absolutely continuous, we obtain K_ν from (21) and by a direct calculation from (20) as

$$K_3 = 0, \qquad K_4 = -\frac{7}{5760}$$

and these imply that

$$\lim_{n\to\infty} n^3 R(f; CM_n) = \lim_{n\to\infty} n^3 R_n^{(3)}(f) = 0$$

and

$$\lim_{n\to\infty} n^4 R(f; CM_n) = \lim_{n\to\infty} n^4 R_n^{(4)}(f) = -\frac{7}{5760}(b-a)^4[f'''(b) - f'''(a)].$$

Example 3. For the corrected Simpson rule, we see that $\alpha = -\frac{1}{60}$ and it has degree of precision 5 ($r = 6$). Thus for any function f such that $f^{(\nu-1)}$ ($\nu = 3, 4, 5, 6$) is continuous of bounded variation, we get C_ν by direct calculations from (18) as

$$C_3 = \frac{7}{20\,250} + \frac{19\sqrt{19}}{81\,000}, \qquad C_4 = \frac{1}{5760}, \qquad C_5 = \frac{1}{58\,320}, \qquad C_6 = \frac{1}{230\,400}$$

and, for any function f such that $f^{(\nu-1)}$ ($\nu = 3, 4, 5, 6$) is absolutely continuous, we obtain K_ν from (21) and by a direct calculation from (20) as

$$K_3 = K_4 = K_5 = 0, \qquad K_6 = \frac{1}{604\,800}$$

and these imply that

$$\lim_{n\to\infty} n^\nu R(f; CS_n) = \lim_{n\to\infty} n^\nu R_n^{(\nu)}(f) = 0, \quad \text{for } \nu = 3, 4, 5,$$

and

$$\lim_{n\to\infty} n^6 R(f; CS_n) = \lim_{n\to\infty} n^6 R_n^{(6)}(f) = \frac{1}{604\,800}(b-a)^4[f^{(5)}(b) - f^{(5)}(a)].$$

Remark 6. It should be mentioned that the same estimates of error for the corrected Simpson rule were obtained in [6]. Furthermore, estimates of error for both classical and corrected trapezoidal rules were obtained in [10] and for the midpoint and corrected midpoint rules in [11].

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